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THE JOINING OF LOCAL EXPANSIONS IN THE THEORY OF NON-LINEAR OSCILLATIONS*

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The behaviour of normal modes of oscillation in non-linear conservative systems with a finite number of degrees of freedom, when the amplitude changes from zero to infinity is studied. In the non-linear case, the normal oscillations represent a generalization of the normal oscillations of linear conservative systems (see /1/). It is assumed that the potential of a non-linear system is a polynomial of even degree in all positional variables. One can construct the trajectories of the normal oscillations in configuration space both for sufficiently small amplitude (a quasi-linear expansion), and for sufficiently large amplitude, using the fact that in these cases the system is close to a uniform system (see /2, 3/). The local expansions obtained are joined using rational-fractional Padé representations (see /4/) which enables the behaviour of oscillation modes to be followed when the amplitude changes continuously.

1. An initial conservative system is defined by the following equations of motion

$$z_i^{\cdot\cdot} + \Pi_{z_i}(z_1, z_2, \ldots, z_n) = 0 \ (i = 1, 2, \ldots, n) \tag{1.1}$$

where the potential $\Pi(z_1, z_2, \ldots, z_n)$ is a positive definite polynomial in $z_1, \ldots, z_n$ whose lowest degree is two, and the highest is $2m$. Here and below we assume that the kinetic energy is reduced to the form $T = {}^1/_2 (z_1^{\cdot 2} + \ldots + z_n^{\cdot 2})$. An equation of this type is often encountered problems of the oscillations of non-linear elastic systems.

After separating one of the coordinates, say z_1, we use the change $z_i = cx_i$, where $c = z_1(0)$. Clearly, $x_1(0) = 1$. In addition, we can assume without loss of generality that $x_1^{\cdot}(0) = 0$. Eqs.(1.1) can be rewritten as follows:

$$x_i^{\cdot\cdot} + V_{x_i}(c, x_1, x_2, \ldots, x_n) = 0, \quad V = \sum_{k=0}^{2m-2} C^k V^{(k-2)}(x_1, x_2, \ldots, x_n) \tag{1.2}$$

where $V^{(r)}$ contains r-th degree terms with respect to the variables in the potential $V(c, x_1, x_2, \ldots, x_n) = \Pi(z_1(x_1), z_2(x_2), \ldots, z_n(x_n))$. Here the energy integral has the form

$$\sum_{i=1}^{n} x_i^{\cdot 2} + V(c, x_1, x_2, \ldots, x_n) = h \tag{1.3}$$

where h is the energy of the system. Henceforth, we shall assume that the oscillation amplitude $c = z_1(0)$ is an independent parameter, and the energy is given by (1.3). Therefore, it is convenient to represent the energy h as the sum of terms corresponding to the uniform components of the potential V,

$$h = \sum_{k=2}^{2m-2} c^k h_k \tag{1.4}$$

On introducing a new independent variable $x \equiv x_1$ and eliminating time from Eqs.(1.2) using the energy integral (1.3), we obtain equations for determining the trajectories $x_i = x_i(x)$ in the configuration space,

$$2x_i''[h - V(c, x, x_2, \ldots, x_n)] + \Big[1 + \sum_{i=2}^{n} (x_i')^2\Big] \times \\ [-x_i' V_x(c, x, \ldots, x_n) + V_{x_i}(c, x, \ldots, x_n)] = 0 \\ (i = 2, 3, \ldots, n) \tag{1.5}$$

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(the prime denotes differentiation with respect to x).

The above equations can be used to find the trajectories of the normal oscillations of the initial system in the form of single-valued analytic functions $x_i = x_i(x)$ (see Fig.1).

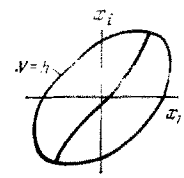

Fig. 1

The trajectories have cusps on the maximum isoenergetic surface $V = h$, where all velocities vanish. However, these points are the singular points of Eqs.(1.5). It can be verified by direct calculation that these singular points, which are the points of intersection of the trajectories with the surface $V = h$, are removable if the trajectory intersects the surface orthogonally, /1 - 3/, that is

$$[-x_i' V_x(c, x, x_2(x), \ldots, x_n(x)) + V_{x_i}(c, x, x_2(x), \ldots, x_n(x))]|_{x=X} = 0 \tag{1.6}$$

Here X is the value of the variable x on the surface $V = h$. As indicated above, one of these values is $X_{(1)} = x(0) = 1$.

For small amplitudes c, for a generating system we must choose a uniform linear system with potential $V^{(2)}$, and for large amplitude a uniform non-linear system with potential $V^{(2m)}$. Both linear and non-linear uniform systems are capable of rectilinear normal oscillations of the form $x_i = k_i x$, the constants k_i being determined from the algebraic equations $k_i V_x^{(r)}(1, k_2, \ldots, k_n) + V_{x_i}^{(r)}(1, k_2, \ldots, k_n) = 0$ (see /1/). The number of these oscillations may exceed the number of degrees of freedom.

For small c, in the vicinity of a linear system we can find trajectories of the normal oscillations in the form of series in powers of x and c,

$$x_i^{(1)} = \sum_{j=0}^{\infty} \alpha_j^{(i)}(x) c^j \equiv \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \alpha_{jl}^{(i)} x^l c^j \quad (i = 2, 3, \ldots, n) \tag{1.7}$$

and in the vicinity of a uniform non-linear system (for large c) in the form of series in powers of x and c^{-1} (see /2 -3/),

$$x_i^{(2)} = \sum_{j=0}^{\infty} \beta_j^{(i)}(x) c^{-j} \equiv \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \beta_{jl}^{(i)} x^l c^{-j} \quad (i = 2, 3, \ldots, n) \tag{1.8}$$

Note that the functions $\alpha_j(x)$ and $\beta_j(x)$ can also be obtained in quadratures since the equations in variations for the normal oscillations of uniform systems reduce to hypergeometric systems /5/.

The constraints on the generating systems both in the linear and the non-linear case are shown in /2, 3/ and they mean that the cases of inner resonance (branching of the normal oscillation) are excluded from the study.

The amplitude values $x = 1$, $x_i^{(1)}(1)$, or $x_i^{(2)}(1)$ (for $x' = x_i' = 0$) fully define the mode of normal oscillations. Therefore, to simplify the calculations we shall consider below only the expansions $\rho_i^{(1)} = x_i^{(1)}(1)$ and $\rho_i^{(2)} = x_i^{(2)}(1)$ in powers of c which are obtained from (1.7) and (1.8) for $x = 1$ (here $\alpha_j^{(i)} = \alpha_j^{(i)}(1)$, $\beta_j^{(i)} = \beta_j^{(i)}(1)$; further, we make use of the abbreviated notation $\sum_{j=0}^{r} \alpha_j^{(i)} c^{\pm j} = \sum \alpha_{i(r)}^{(\pm j)}$, and analogous notation for other similar schemes):

$$\rho_i^{(1)} = \sum \alpha_{i(\infty)}^{(j)}, \quad \rho_i^{(2)} = \sum \beta_{i(\infty)}^{(-j)} \tag{1.9}$$

2. To join the local expansions (1.9), and to study the behaviour of the normal oscillation trajectories for arbitrary values of the amplitude c we use the rational-fractional Pade diagonal representations

$$P_s^{(i)} = \frac{\Sigma a_{i(s)}^{(j)}}{\Sigma b_{i(s)}^{(j)}} \quad (s = 1, 2, 3, \ldots; \; i = 2, 3, \ldots, n) \tag{2.1}$$

In addition to these representations in positive powers of c, we also make use of the representation in negative powers of c, by multiplying the numerator and denominator of (2.1) by c^{-s} and comparing the expressions obtained and Eq.(2.1) with expansion (1.9). This yields

$$(\Sigma\alpha_{i(\infty)}^{(j)})(\Sigma b_{i(s)}^{(j)}) = \Sigma a_{i(s)}^{(j)}, \quad (\Sigma\beta_{i(\infty)}^{(-j)})(\Sigma b_{i(s)}^{(j-s)}) = \Sigma a_{i(s)}^{(j-s)} \tag{2.2}$$

Retaining the terms of order c^r $(-s \leqslant r \leqslant s)$ and comparing the coefficients of identical powers of c, we obtain $n-1$ systems of $2(s+1)$ linear algebraic equations for determining $a_j^{(i)}$, $b_j^{(i)}$ $(j = 0, 1, 2, \ldots, s)$. The determinants of these systems have the form

$$\Delta_s^{(i)} = \det A_s^{(i)} \tag{2.3}$$

$$A_s^{(i)} = \left\| \begin{array}{c|c} -I_{s+1} & B_{\alpha s+1}^{(i)} \\ \hline -I_{s+1} & B_{\beta s+1}^{(i)T} \end{array} \right\|, \qquad B_{\gamma s+1}^{(i)} = \left\| \begin{array}{cccc} \gamma_0 & 0 & \ldots & 0 \\ \gamma_1 & \gamma_0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ \gamma_s & \gamma_{s-1} & \ldots & \gamma_0 \end{array} \right\|$$

where $s+1$ is the dimension of the matrix, I_{s+1} is the unit matrix, and T is the symbol of transposition.

Since the determinants (2.3) are not, generally speaking, zero, the systems of algebraic equations have a unique trivial exact solution $a_j^{(i)} = b_j^{(i)} = 0$.

Let us separate a certain approximate Padé representation which corresponds to the retained terms in expansions (1.9), with non-zero coefficients $a_j^{(i)}$ and $b_j^{(i)}$. We assume that $b_0^{(i)} \neq 0$ (otherwise $\rho_i^{(1)}(1) \to \infty$ as $c \to 0$). We can assume without loss of generality that $b_0^{(i)} = 1$. Now the systems of algebraic equtions for finding $a_j^{(i)}$ and $b_j^{(i)}$ become overdetermined. All unknown coefficients $a_0^{(i)}, a_1^{(i)}, \ldots, a_s^{(i)}, b_1^{(i)}, b_2^{(i)}, \ldots, b_n^{(i)}$ $(i = 2, 3, \ldots, n)$ are determined from the $2s+1$ equations, and the "discrepancy" of this approximate solution is obtained by substituting the values of all the coefficients into the remaining equation. Clearly, the discrepancy is determined by the value of $\Delta_s^{(i)}$, since for $\Delta_s^{(i)} = 0$ we obtain non-zero solutions and, correspondingly, the exact Padé representations for expansions (1.9) with the given approximation with respect to c.

Hence follows the necessary condition regarding the convergence of the sequence of Padé approximations (2.1) to the rational fractional functions

$$P^{(i)} = \Big(\sum_{j=0}^{\infty} a_j^{(i)} c^j \Big) \Big(\sum_{j=0}^{\infty} b_j^{(i)} c^j \Big)^{-1} \quad (b_0^{(i)} \equiv 1) \tag{2.4}$$

as $s \to \infty$. It is

$$\lim_{s\to\infty} \Delta_s^{(i)} = 0 \quad (i = 2, 3, \ldots, n) \tag{2.5}$$

In fact, if conditions (2.5) are not satisfied then obviously the non-zero values of the coefficients $a_j^{(i)}$ and $b_j^{(i)}$ are not obtained in representations (2.4).

We note that the limit Padé representations $P^{(i)}$ will be useful for describing a solution for any value of c, if the functions $P^{(i)}$ have no poles.

Since in general there are several local quasilinear expansions (1.7), and essentially non-linear local expansions (1.8), and the numbers of these expansions may not be the same, the convergence conditions (2.5) enable us to establish the connection between qusilinear and the essentially non-linear expansions, i.e. to determine which of them correspond to one solution, and which to various others.

3. For a concrete analysis, using the technique described, let us examine a conservative system with two degrees of freedom, whose potential contains terms in the second and the fourth power of the variables. On making the replacement $z_1 = cx, z_2 = cy$, where $c = z_1(0)$, $(x(0) = 1)$, we obtain

$$V = d_1 \frac{x^2}{2} + d_2 \frac{y^2}{2} + d_3 xy + c^2 \Big(\gamma_1 \frac{x^4}{4} + \gamma_2 x^3 y + \gamma_3 \frac{x^2 y^2}{2} + \gamma_4 x y^3 + \gamma_5 \frac{y^4}{4} \Big) \equiv V^{(2)} + c^2 V^{(4)}$$

Here the equation for determining the trajectory $y(x)$ has the form

$$2y''(h - V) + (1 + y'^2)(-y' V_x + V_y) = 0 \tag{3.1}$$

and the boundary conditions (1.6) can be expressed as

$$(-y' V_x + V_y)_{h-V=0} = 0$$

On the surface $h - V = 0$ (when $x' = y' = 0$) one of the values of x is $x(0) = 1$; therefore the corresponding boundary condition has the form

$$(-y' V_x + V_y)|_{x=1} = 0 \tag{3.2}$$

The system is symmetric about the origin, and therefore the second boundary condition (for $x = -1$) is the same as (3.2).

Since the trajectory will be represented in the form of expansions in powers of c^2 or c^{-2}, we introduce a parameter ν ($\nu = c^2$ in the quasilinear case, and $\nu = c^{-2}$ in the essentially non-linear case). Now the solution of Eq. (3.1) is sought in the form of series in the small parameter ν,

$$y = \sum_{j=0}^{\infty} y_j(x) \nu^j$$

In Eqs. (3.1) and the boundary conditions (3.2), $V = V_0 + \nu V_1$, $h = h_0 + \nu h_1$; at the same time in the quasilinear case

$$\nu = c^2,\ V_0 = V^{(2)},\ V_1 = V^{(4)},\ h_0 = V^{(2)}|_{x=1},\ h_1 = V^{(4)}|_{x=1}$$

and in the essentially non-linear case

$$\nu = c^{-2},\ V_0 = V^{(4)},\ V_1 = V^{(2)},\ h_0 = V^{(4)}|_{x=1},\ h_1 = V^{(2)}|_{x=1}$$

In the zeroth-order approximation with respect to ν both linear and non-linear systems are capable of the rectilinear normal mode of oscillations in the form $y = k_0 x$. The constants k_0 are determined from algebraic equations of the second degree (linear system) or the fourth degree (non-linear system), which are obtained from (3.1),

$$-k_0 V_{0x}(1, k_0) + V_{0y}(1, k_0) = 0 \tag{3.3}$$

To be specific, let $d_1 = d_2 = 3$; $d_3 = -2$; $\gamma_1 = 1$; $\gamma_2 = 0$; $\gamma_3 = 3$; $\gamma_4 = 0.2091$; $\gamma_5 = 2$. We will write the equations of motion for this system

$$\begin{aligned} &x'' + x + 2(x - y) + c^2(x^3 + 3xy^2 + 0.2091\ y^3) = 0 \\ &y'' + y + 2(y - x) + c^2(2y^3 + 3x^2y + 0.6273y^2x) = 0 \end{aligned} \tag{3.4}$$

In the linear limit ($\nu = 0$), using Eqs.(3.3) we obtain two rectilinear normal modes of oscillations of the form $y = k_0 x$, $k_0^{(1)} = 1$; $k_0^{(2)} = -1$, and the non-linear system (the equations of motion contain the cubic terms in x, y only) admits of four such forms: $k_0^{(3)} = 1.496$; $k_0^{(4)} = 0$; $k_0^{(5)} = -1.279$; $k_0^{(6)} = -5$.

To determine the curvilinear trajectories of normal oscillations close to straight lines, we use Eqs.(3.1) and the boundary conditions (3.2). In particular, to a first approximation with respect to ν the equation for determining the trajectory has the form

$$2y_1''(h_0 - V_0) + (1 + k_0^2)[-y_1'V_{0x} - y_1(V_{0yy} - k_0 V_{0xy}) - k_0 V_{1x} + V_{1y}] = 0$$

where we substitute $y = k_0 x$ everywhere in the functions V_0 and V_1 and in their derivatives. The equations in the subsequent approximations with respect to ν are obtained similarly. Such a splitting should also be carried out in the boundary conditions (3.2).

On retaining in the solution terms containing x and x^3 (there are no terms in power of x^2 because of the system's symmetry about the origin), and performing a calculation in two approximations with respect to ν, we obtain (henceforth, everywhere $\rho = y(1)$):

in the quasilinear case ($\nu = c^2$)

$$\begin{aligned} &y^{(1)} = x + \nu(-0.532\ x + 0.355\ x^3) + \nu^2\ (1.970x - 2.405\ x^3) \\ &y^{(2)} = -x + \nu(-0.099\ x - 0.013\ x^3) + \nu^2(0.041\ x + 0.009x^3) \\ &\rho^{(1)} = 1 - 0.177\nu - 0.435\nu^2 \quad \rho^{(2)} = -1 - 0.112\nu + 0.050\ \nu^2 \end{aligned}$$

in the essentially non-linear case ($\nu = c^{-2}$)

$$\begin{aligned} &y^{(3)} = 1.496\ x + \nu(0.830x - 0.098x^3) + \nu^2(0.822x - 0.123x^3) \\ &y^{(4)} = \nu(x + 0.667x^3) + \nu^2(1.129x - 4x^3) \\ &y^{(5)} = -1.279x + \nu(0.844x - 0.077x^3) + \nu^2(-2.624x - 0.351x^3) \\ &y^{(6)} = -5x + \nu(-2.844x - 0.321\ x^3) + \nu^2(1.962x - 0.099\ x^3) \\ &\rho^{(3)} = -1 - 0.177\nu - 0.435\nu^2 \quad \rho^{(4)} = 1.667\nu - 2.871\nu^2 \\ &\rho^{(5)} = -1.279 + 0.767\nu - 2.273\nu^2 \quad \rho^{(6)} = -5 - 3.165\nu + 1.863\ \nu^2 \end{aligned} \tag{3.5}$$

Comparing pairwise the quasilinear expansions $\rho^{(j)}$ ($j = 1, 2$) with the expansions for large amplitudes $\rho^{(j)}$ ($j = 3, 4, 5, 6$), we compute for each pair the determinant (2.4) for $s = 1, 2$. The calculations show that the error decreases as s increases only for pairs $\rho^{(1)}$ and $\rho^{(4)}$, and $\rho^{(2)}$ and $\rho^{(5)}$. Thus, we must assume that each pair corresponds to one solution. We join the local expansions using representation (2.1) for $s = 2$.

For pairs $\rho^{(1)}$ and $\rho^{(4)}$ we have the representation

$$\rho = y(1) \approx \frac{1 + 1{,}20c^2}{1 + 1.61c^2 + 0{,}72c^4} \tag{3.6}$$

(the coefficient $b_2 = 0.72$ computed with an error of 0.06). The representation for pairs and $\rho^{(2)}$ is $\rho^{(5)}$

$$\rho = y(1) \approx \frac{-1 - 1.11c^2 - 0.275c^4}{1 + 1.00c^2 + 0.215c^4} \tag{3.7}$$

(the coefficient $b_2 = 0.215$ is computed with an error of 0.01).

Since for the local expansions $\rho^{(4)}$ and $\rho^{(6)}$ there are no corresponding expansions for small amplitudes, it is obvious that as the amplitude decreases these two solutions disappear, merging at a certain limit point. The approximate value of the amplitude c_0 for which this occurs is determined by comparing $\rho^{(3)}$ with $\rho^{(6)}$. We obtain $c = c_0 \approx 0.5$.

Thus, the rational fractional Padé presentation enable us to judge the non-local behaviour of the oscillations of finite-dimensional non-linear systems. Figure 2 shows the evolution

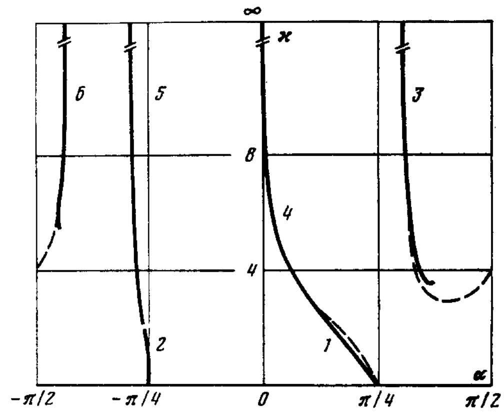

Fig. 2

of the oscillation modes using parameters $\varkappa = \ln(1 + c^2h^2)$, $\alpha = \operatorname{arctg} \rho$, (the graph is periodic with respect to α, with a period of 2π). The numbers $1, 2, \ldots, 6$ mark the curves which correspond to the expansions $\rho^{(1)}, \rho^{(2)}, \ldots, \rho^{(6)}$ in powers of c. The solid lines correspond to an analytic solution (the representations (3.6) and (3.7) were used) and the dashed lines represent the check calculation made by A.L. Zhupiev on a computer. Notice the good agreement between the analytical results and the numerical calculations. For the solution of (2.5), the representation (3.7) and numerical calculations give identical curves.

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